

## EXTERIOR MOTION IN THE RESTRICTED PROBLEM OF THREE BODIES\*

BY

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1. **Introduction.** The study of the restricted problem of three bodies, as distinguished from the general problem, may be said to have been initiated by G. W. Hill in 1877. We find in his *Lunar theory*† exactly the present assumption of one infinitesimal body moving subject to the attraction of two bodies which revolve in circles about their center of mass. He introduced the device of the rotating plane of reference and named and ably employed the integral of Jacobi while proving that the moon's distance from the earth can not exceed a certain figure. The next notable contribution to this subject was the memoir‡ of Poincaré crowned in 1899 by King Oscar of Sweden. Although considering primarily a very general class of dynamic problems, he applies his theory particularly to the restricted problem of three bodies. Poincaré later elaborated this memoir into his famous work *Les Méthodes Nouvelles de la Mécanique Céleste*. In 1897 Sir George Darwin in his memoir on *Periodic orbits*§ considered the restricted problem of three bodies in the plane, elaborated Hill's discussion of the curves of zero relative velocity, and by direct numerical calculations rendered practically certain the existence of periodic orbits of certain classes. In this he took no account of the work of Poincaré.

The new and powerful but comparatively difficult methods of attack originated by Poincaré were employed by G. D. Birkhoff in 1914.|| By representing the state of motion of the particle at any instant by a point in a space of higher dimensions Birkhoff reduces the discussion of the orbits to a problem in analysis situs in this space. The most striking result is the proof of the existence of certain periodic orbits within the closed oval of zero relative velocity about either of the two heavy bodies. F. R. Moulton and others also followed lines pointed out by Poincaré in the discussion of periodic orbits.¶

In the memoirs thus far mentioned comparatively little attention is devoted to motion of the particle in distant portions of the plane. However in

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† *Researches on the lunar theory*, American Journal of Mathematics, vol. 1 (1878), p. 5.

‡ *Sur le problème des trois corps*, Acta Mathematica, vol. 13 (1890), p. 1.

§ Acta Mathematica, vol. 21 (1897), p. 99.

|| *The restricted problem of three bodies*, Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915), p. 265.

¶ *Periodic Orbits*, Publications of the Carnegie Institution of Washington, No. 161, 1920.

1927 B. O. Koopman published a paper\* opening the field of research on motion of the particle outside of the closed outer oval of zero relative velocity, and treating especially orbits extending to infinity. He was able to show that with comparatively slight alterations in Birkhoff's methods the major portion of the latter's conclusions concerning interior motion may be proved for exterior motion also. It is the purpose of the present paper to examine more in detail certain aspects of this exterior motion, confining the treatment to cases in which the outer oval is closed, but placing no additional restriction on the constant of Jacobi nor on the ratio of the two finite masses.

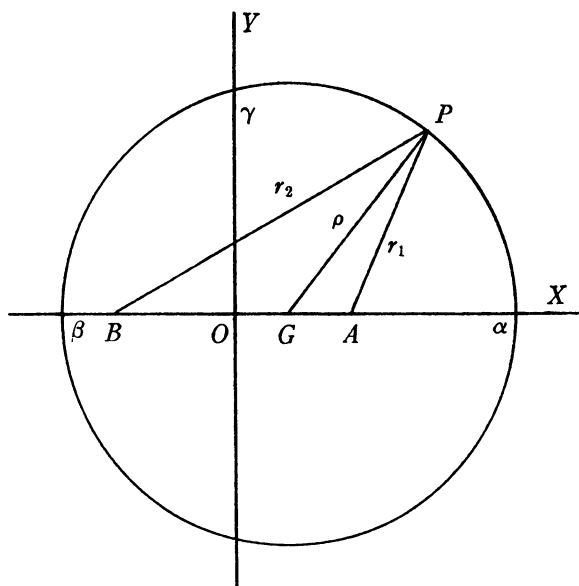
We shall first in §2 briefly develop the equations of motion and their one possible integral. In §3 we carry out a careful examination of the outer ovals of zero relative velocity. The salient result of this section is a necessary and sufficient condition that the outer oval be closed, both for a given ratio of the two finite masses and also independently of this ratio. §4 is devoted to a study of the properties of the force function and their reduction to inequalities, possibly of no great interest in themselves but essential to the theorems to follow. §5 is a study of the exterior orbits in the neighborhood of their points of contact with the closed outer oval. The fact that some of the positions of equilibrium in the rotating plane, while constituting limiting cases of the class of orbits studied, still do not possess the same properties as these orbits forces the inequalities here employed to be extremely close. In §6 we develop two different sets of sufficient conditions that the particle recede to infinity and extend Koopman's discussion of the behavior of the areal velocity in fixed space for distant portions of the plane. §7 contains four theorems concerning the angular velocity of the particle in the fixed and rotating plane. Perhaps the most striking result here is that for orbits not extending to infinity the motion in the rotating plane can be direct only within a narrow ring surrounding the closed outer oval, the width of the ring approaching zero as its diameter increases. §8 discusses the total angular displacements in the fixed and rotating plane, showing for instance in Theorem XII that the infinitesimal body can never advance as much as one radian in the rotating plane. Two related theorems and corollaries complete the section.

2. **The equations of motion; Jacobi's integral.** In the restricted problem of three bodies the hypothesis is made that two of the bodies move subject to the law of gravitation in concentric coplanar circles about their common center of mass with a constant angular velocity, while a third body moves in this plane subject to their attractions but without affecting their motion.

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\* *On rejection to infinity and exterior motion in the restricted problem of three bodies*, these *Transactions*, vol. 29 (1927), p. 287.

The study of the motion of this third or infinitesimal body is facilitated by two devices. In the first place, the sum of the masses of the two finite bodies is chosen as the unit of mass, and the distance between their centers of mass is chosen as the unit of distance, while the time they require to sweep out a unit angle is chosen as the unit of time. If these units be employed the constant of gravitation is also unity. Secondly, the motion of the infinitesimal body is referred to a rotating plane lying in the plane of the motion and rigidly attached to the two finite bodies. Their coördinates thus enter the equations of motion of the third body only implicitly.



In this moving plane we shall employ two coördinate systems. The first is a rectangular Cartesian system having the  $X$  axis on the line of centers of the two finite masses and the origin at their *midpoint*  $O$ . In this system the larger of the two finite bodies having a mass of  $\frac{1}{2} + k$  ( $0 \leq k \leq \frac{1}{2}$ ) is fixed at the point  $A$  ( $\frac{1}{2}, 0$ ) and the smaller body of mass  $\frac{1}{2} - k$  is fixed at the point  $B$  ( $-\frac{1}{2}, 0$ ). For a counter clockwise sense of rotation of the finite bodies the equations of motion of the infinitesimal body in the rotating plane may be shown to be

$$(1) \quad x'' - 2y' - (x - k) = \frac{\partial M}{\partial x},$$

$$(2) \quad y'' + 2x' - y = \frac{\partial M}{\partial y},$$

where  $M(x, y)$  is the force function

$$M(x, y) = \frac{\frac{1}{2} + k}{r_1} + \frac{\frac{1}{2} - k}{r_2}, \quad r_1 = \{(x - \frac{1}{2})^2 + y^2\}^{1/2}, \quad r_2 = \{(x + \frac{1}{2})^2 + y^2\}^{1/2},$$

and where the primes denote differentiation with respect to the time. The second coördinate system which we shall employ in the rotating plane is a polar system having its initial line along the line of centers of mass of the two finite bodies, but its pole at their *center of mass*  $G$ . The Cartesian coördinates of the pole of the polar system are thus  $G(k, 0)$ . In this polar system the equations of motion of the infinitesimal body in the rotating plane are

$$(3) \quad \rho'' - \rho(1 + \theta')^2 = \frac{\partial M}{\partial \rho},$$

$$(4) \quad \rho\theta'' + 2\rho\rho'(1 + \theta') = \frac{\partial M}{\partial \theta},$$

where  $M(\rho, \theta)$  is the force function as before, and where

$$r_1 = \{\rho^2 + (\frac{1}{2} - k)^2 - 2(\frac{1}{2} - k)\rho \cos \theta\}^{1/2},$$

$$r_2 = \{\rho^2 + (\frac{1}{2} + k)^2 + 2(\frac{1}{2} + k)\rho \cos \theta\}^{1/2}.$$

The equations (1), (2) or (3), (4) admit the integral first pointed out by Jacobi\*

$$(5) \quad x'^2 + y'^2 = \rho'^2 + \rho^2\theta'^2 = \rho^2 + 2M - C.$$

This is known as the integral of Jacobi and the constant  $C$  as Jacobi's constant.

**3. The outer ovals.** It will be observed that the first two members of equations (5) are expressions for the square of the velocity of the infinitesimal body relative to the rotating plane. If we introduce a point function

$$\Omega = \frac{1}{2}\rho^2 + M = \frac{1}{2}\rho^2 + \frac{\frac{1}{2} + k}{r_1} + \frac{\frac{1}{2} - k}{r_2}$$

the curves of zero relative velocity have the equation

$$v^2 = 2\Omega - C = 0.$$

These curves have forms varying with  $k$  and  $C$  in a well known fashion, and for sufficiently large values of  $C$  there will always be one branch of the curve forming a single outer oval surrounding both finite bodies. We shall say that

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\* Comptes Rendus de l'Académie des Sciences de Paris, vol. 3 (1836), p. 59.

this outer oval is closed\* if the value of  $2\Omega - C$  changes from negative to positive as one passes across the oval anywhere on it in the sense of increasing  $\rho$ , and we shall say that the infinitesimal body is performing exterior motion if it moves outside of or on such a closed outer oval.

Let us examine the conditions under which such an oval may exist. It will be proved in §4 that for a fixed  $k$  the force function  $M$  attains its maximum value on a given circle  $\rho = \rho_0$  ( $\rho_0 > \frac{1}{2} + k$ ) at the point  $\beta$  where this circle cuts the negative  $X$  axis. This statement must clearly also hold for  $2\Omega = \rho^2 + 2M$ . Along the negative  $X$  axis we have

$$\frac{\partial \Omega}{\partial \rho} = \rho - \frac{\frac{1}{2} + k}{(\rho - k + \frac{1}{2})^2} - \frac{\frac{1}{2} - k}{(\rho - k - \frac{1}{2})^2}$$

and it is easily found that this quantity has exactly one zero for  $\rho > \frac{1}{2} + k$ . Let us designate by  $\rho_0$  this value of  $\rho$  which thus corresponds to the absolute minimum of  $2\Omega$  on the segment of the negative  $X$  axis considered, and let us designate this minimum value of  $2\Omega$  by  $C_0$ .

**THEOREM I.** *A necessary and sufficient condition that the curve  $2\Omega - C = 0$  possess a branch constituting a closed outer oval is that  $C > C_0$ .*

First, the condition is necessary, for suppose that a closed outer oval were to exist for  $C < C_0$ . Such an oval to include the two finite bodies would have to cross the negative  $X$  axis outside of the point  $B$  and at this crossing point we would therefore have  $\rho > \frac{1}{2} + k$ . But at this point the value of  $2\Omega$  would be  $C$ , a value less than the previously established minimum of  $2\Omega$  in this interval. A contradiction is thus established and there could be no closed outer oval for  $C < C_0$ . It is also readily seen that with the above agreement as to what constitutes a closed oval there could not be one for  $C = C_0$ .

Second, the condition is sufficient. On the negative  $X$  axis as  $\rho$  increases from  $\rho_0$  the quantity  $2\Omega$  continually increases from  $C_0$ , and since  $2\Omega$  is here continuous and becomes arbitrarily great it assumes once and only once any assigned value greater than  $C_0$ . Let us designate by  $\rho_1$  ( $\rho_1 > \rho_0$ ) the value of  $\rho$  for which  $2\Omega$  takes on in the above interval a designated value  $C > C_0$ ,

$$\rho_1^2 + \frac{1 + 2k}{\rho_1 - k + \frac{1}{2}} + \frac{1 - 2k}{\rho_1 - k - \frac{1}{2}} = C.$$

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\* As an explanation for the choice of the above special definition of what shall constitute a "closed" outer oval we may point out that if the oval were merely closed in the usual sense of a closed Jordan curve it might have upon it under certain conditions one or two of the positions of equilibrium giving the Lagrangian solutions. These special solutions would constitute actual exceptions to several of the theorems we wish to prove and render the proofs of others somewhat awkward. The above definition excludes these exceptional cases without materially diminishing the range of application of the ensuing discussion.

Since  $C > C_0$  the quantity  $2\Omega - C$  must be negative for  $\rho = \rho_0$  on the negative  $X$  axis and a fortiori negative everywhere on the circle  $\rho = \rho_0$  since, for a given  $\rho$ ,  $2\Omega$  has its maximum value on the negative  $X$  axis. Also  $2\Omega - C$  is clearly positive everywhere on the circle  $\rho = C^{1/2}$ . Hence if a moving point start at any point on the circle  $\rho = \rho_0$  and remaining thereafter outside of this circle pass continuously to any point on the circle  $\rho = C^{1/2}$ , the value of  $2\Omega - C$  at this moving point must change continuously from negative to positive and the moving point must cross the curve  $2\Omega - C = 0$  at least once. The curve thus possesses a branch constituting an oval closed in the usual sense which runs completely around in the ring between the two concentric circles  $\rho = \rho_0$  and  $\rho = C^{1/2}$ . To see that this oval is closed in the special sense of the definition, i.e. that  $2\Omega - C$  always changes from negative to positive as one passes outward across the oval, it will evidently suffice to know that  $\partial\Omega/\partial\rho$  is everywhere positive for  $\rho > \rho_0$ . This last fact will be proved in §4. The condition  $C > C_0$  is thus also sufficient for the existence of the closed outer oval.

The above argument fails in the case  $k = \frac{1}{2}$  at the point where we establish that  $\partial\Omega/\partial\rho$  possesses a zero within the interval  $\rho > \frac{1}{2} + k$  of the negative  $X$  axis. In fact, however, in this case we have

$$2\Omega = \rho^2 + \frac{2}{\rho}, \quad \frac{\partial\Omega}{\partial\rho} = \rho - \frac{1}{\rho^2}, \quad \rho_0 = 1, \quad C_0 = 3.$$

Our curve  $2\Omega - C = 0$  becomes

$$\rho^3 - C\rho + 2 = 0.$$

The discriminant of this reduced cubic is  $4(C^3 - 27)$ , so that for  $C > C_0$  there are three real distinct roots, only one of which, however, is greater than  $\rho_0$ . The outer oval then exists, being the circle  $\rho = \rho_1$  where  $\rho_1$  is the root of the above equation greater than  $\rho_0$ . It is clear that  $\partial\Omega/\partial\rho > 0$  everywhere on it and it is therefore closed in the special sense of the definition. The condition thus holds for this special case.

To apply the above criterion for the existence of the closed oval to any given orbit we must compare the value of  $C$  for the orbit with the value of  $C_0$  as above defined. The actual computation of  $C_0$  is of course done by solving the equation

$$(a) \quad \rho_0 - \frac{\frac{1}{2} + k}{(\rho_0 - k + \frac{1}{2})^2} - \frac{\frac{1}{2} - k}{(\rho_0 - k - \frac{1}{2})^2} = 0$$

for its only root  $\rho_0 \geq \frac{1}{2} + k$  and then substituting this value of  $\rho_0$  in the equation

$$(b) \quad C_0 = \rho_0^2 + \frac{1 + 2k}{\rho_0 - k + \frac{1}{2}} + \frac{1 - 2k}{\rho_0 - k - \frac{1}{2}}.$$

The values of  $C_0$  for the different values of  $k$  differ only slightly, ranging, as we shall see, from 3.00 to 3.56. For this reason it will often be unnecessary to compute the value of  $C_0$  in the individual case since if the  $C$  of the given orbit be greater than the greatest value  $\bar{C}_0$  assumed by  $C_0$  for all values of  $k$  we shall be assured of the existence of the closed outer oval regardless of the value of  $k$  involved. We recall that  $C_0$  is a function of  $\rho_0$  and  $k$ , but that  $\rho_0$  is a single-valued, continuous function of  $k$  and hence  $C_0$  may be regarded as a function of  $k$  only. We may therefore write as a necessary condition for a relative extremum of  $C_0$

$$\frac{dC_0}{dk} = \frac{\partial C_0}{\partial k} + \frac{\partial C_0}{\partial \rho_0} \frac{d\rho_0}{dk} = 0.$$

But the equation (a) by which  $\rho_0$  is determined as a function of  $k$  is exactly

$$\frac{1}{2} \frac{\partial C_0}{\partial \rho_0} = 0,$$

so that we have finally

$$(c) \quad \frac{\partial C_0}{\partial k} = \frac{4k^2 - 4k\rho_0 + 1}{(\rho_0 - k + \frac{1}{2})^2(\rho_0 - k - \frac{1}{2})^2} = 0.$$

This excludes the possibility of  $k$  being zero at an extremum of  $C_0$  and yields

$$(d) \quad \rho_0 = k + \frac{1}{4k}.$$

On substituting in equation (a) we find

$$(e) \quad k^4 + 4k^3 - 1/16 = 0.$$

The only positive root of this equation is  $k_0 = 0.24509302 \dots$  and this falls within the permitted range for  $k$ . Thus the extremum, if it exists, is unique. A somewhat lengthy but elementary calculation which we shall omit shows that the corresponding value of  $C_0$  is in fact a relative maximum and since  $C_0$  has no other extrema in the interval considered this relative maximum must be the absolute maximum  $\bar{C}_0$ . In form for computation our equations are

$$k_0^4 + 4k_0^3 - \frac{1}{16} = 0, \quad \rho_0 = k_0 + \frac{1}{4k_0}, \quad C_0 = \rho_0^2 + 8k_0 = 2k_0^2 + 12k_0 + \frac{1}{2},$$

which yield

$$k_0 = 0.24509302 \dots, \quad \rho_0 = 1.2651139 \dots, \quad C_0 = 3.5612574 \dots$$

In conclusion we may therefore state

**THEOREM II.** *A necessary and sufficient condition that a closed outer oval exist regardless of the ratio of the two finite masses is that  $C > \bar{C}_0 = 3.5612574 \dots$*

It will also be desirable to have in mind the least values of  $C_0$  and  $\rho_0$  for which a closed outer oval exists. The least value of  $C_0$  regarded as a function of  $k$  must occur for  $k$  at one end or the other of its range, and we find by direct computation that  $k = \frac{1}{2}$  gives the lesser value which is 3. Hence we have  $C_0 \geq 3$ . We have seen that for  $k = \frac{1}{2}$ ,  $\rho_0 = 1$  and evidently this is the least value of  $\rho_0$ , for the result of substituting 1 for  $\rho_0$  in the first member of equation (a) is negative and the root  $\rho_0$  must be greater than 1. We should also bear in mind for future reference the fact that the least value of  $\rho$  on a closed outer oval is the value  $\rho_1$  assumed by  $\rho$  at the point where the oval crosses the negative  $X$  axis and that we have  $\rho_1 > \rho_0 \geq 1$ . In other words if  $\rho$  has a certain value at any exterior point then the point of the negative  $X$  axis having this same value of  $\rho$  is also an exterior point.

**4. Properties of the force function.** The present section is devoted to a detailed study of the force function  $M$  and its derivatives. The interpretation of the results of this calculation is reserved for the succeeding sections of this paper.

Let us fix  $k$  at any value in its range  $0 \leq k \leq \frac{1}{2}$  and consider the variation of the force function  $M$  and the associated function  $N$ ,

$$M = \frac{\frac{1}{2} + k}{r_1} + \frac{\frac{1}{2} - k}{r_2}, \quad N = \frac{\frac{1}{2} + k}{r_1^2} + \frac{\frac{1}{2} - k}{r_2^2},$$

for  $\rho > \frac{1}{2} + k$ . Due to their symmetry with respect to the  $X$  axis we may regard these point functions as functions of  $x$  and  $\rho$  only, writing  $r_1$  and  $r_2$  as

$$r_1 = \left\{ \rho^2 + \left( \frac{1}{2} - k \right)^2 - 2 \left( \frac{1}{2} - k \right) (x - k) \right\}^{1/2},$$

$$r_2 = \left\{ \rho^2 + \left( \frac{1}{2} + k \right)^2 + 2 \left( \frac{1}{2} + k \right) (x - k) \right\}^{1/2}.$$

If we now fix  $\rho$  at any value  $\rho_0$  in the above range and allow  $x$  to vary between  $-\rho_0 + k$  and  $+\rho_0 + k$ , we find

$$\frac{\partial M}{\partial x} = \left( \frac{1}{4} - k^2 \right) \left( \frac{1}{r_1^3} - \frac{1}{r_2^3} \right), \quad \frac{\partial N}{\partial x} = 2 \left( \frac{1}{4} - k^2 \right) \left( \frac{1}{r_1^4} - \frac{1}{r_2^4} \right).$$

These derivatives of  $M$  and  $N$  having thus the sign of  $x$ , the quantities  $M$  and  $N$  must have an absolute minimum for  $x=0$ , i.e. at the points  $\gamma$  where our circle  $\rho = \rho_0$  cuts the  $Y$  axis, while likewise  $M$  and  $N$  must have relative maxima at the points  $\alpha$  and  $\beta$  where the circle cuts the positive and negative  $X$  axis respectively. See figure on p. 813. We find by direct computation that



$M$  and  $N$  attain their absolute maxima at  $\beta$ , and on computing these various extrema conclude that

$$(6) \quad \frac{1}{(\rho^2 + \frac{1}{4} - k^2)^{1/2}} \leq M \leq \frac{\frac{1}{2} + k}{\rho - k + \frac{1}{2}} + \frac{\frac{1}{2} - k}{\rho - k - \frac{1}{2}},$$

$$(7) \quad \frac{1}{\rho^2 + \frac{1}{4} - k^2} \leq N \leq \frac{\frac{1}{2} + k}{(\rho - k + \frac{1}{2})^2} + \frac{\frac{1}{2} - k}{(\rho - k - \frac{1}{2})^2}.$$

The above argument fails in the case  $k = \frac{1}{2}$  since in this case  $\partial M/\partial x$  and  $\partial N/\partial x$  vanish identically. But then  $M$  and  $N$  reduce to

$$M = 1/\rho, \quad N = 1/\rho^2,$$

and our conclusions (6) and (7), although trivial, still hold.

As  $\rho$  becomes larger both the maximum and minimum of  $M$  take on values approximating that of  $1/\rho$ . In fact for  $\rho \geq 3^{1/2}$  we may show that

$$(8) \quad \frac{1}{\rho} - \frac{1}{8\rho^3} < M < \frac{1}{\rho} + \frac{1}{4\rho^2}.$$

To prove the first inequality we observe successively that

$$\rho^2 \geq 3 > 1/12, \quad 0 > -3\rho^2 + 1/4, \quad 64\rho^6 > 64\rho^6 - 3\rho^2 + 1/4.$$

The last of these inequalities yields

$$\frac{1}{\rho^2 + \frac{1}{4}} > \left( \frac{1}{\rho} - \frac{1}{8\rho^3} \right)^2,$$

and on comparison with the first of inequalities (6) we find the first of inequalities (8) to follow. On reference to the second of inequalities (6) it is apparent that the second of inequalities (8) will be established if we can show that for  $\rho \geq 3^{1/2}$  we have

$$\frac{\frac{1}{2} + k}{\rho - k + \frac{1}{2}} + \frac{\frac{1}{2} - k}{\rho - k - \frac{1}{2}} < \frac{1}{\rho} + \frac{1}{4\rho^2}.$$

We find the equivalent inequality

$$-\rho^2 + (1 + 2k - 4k^2)\rho + (\frac{1}{4} - k^2) < 0$$

and observe that if the first member is negative for  $\rho = 3^{1/2}$  it will be negative for  $\rho \geq 3^{1/2}$ . But for  $\rho = 3^{1/2}$  the quantity may be written

$$-(1 + 4 \cdot 3^{1/2}) \left( k - \frac{12 - 3^{1/2}}{47} \right)^2 - \frac{469 - 184 \cdot 3^{1/2}}{188},$$

which is evidently always negative. Thus the second of inequalities (8) is proved.

As an immediate consequence of inequality (8) we observe that the total variation in  $M$  for a given  $\rho \geq 3^{1/2}$  must be less than the quantity

$$\frac{1}{4\rho^2} + \frac{1}{8\rho^3} < \frac{1}{3\rho^2},$$

and consequently

$$(9) \quad \frac{\frac{1}{2} + k}{\rho - k + \frac{1}{2}} + \frac{\frac{1}{2} - k}{\rho - k - \frac{1}{2}} - M < \frac{1}{3\rho^2} \quad (\rho \geq 3^{1/2}).$$

We may in a similar way limit the quantity  $\partial M / \partial \rho$ . Referring to the definition of  $M$  we see that

$$-\frac{\partial M}{\partial \rho} = \frac{\frac{1}{2} + k}{r_1^2} \cdot \frac{\rho - (\frac{1}{2} - k) \cos \theta}{r_1} + \frac{\frac{1}{2} - k}{r_2^2} \cdot \frac{\rho + (\frac{1}{2} + k) \cos \theta}{r_2}.$$

Since the second factors of the terms of the second member are, as the reader may easily convince himself, the cosines of the angles  $(r_1, \rho)$  and  $(r_2, \rho)$ , they can not exceed unity and it follows that

$$-\frac{\partial M}{\partial \rho} \leq \frac{\frac{1}{2} + k}{r_1^2} + \frac{\frac{1}{2} - k}{r_2^2}.$$

We recognize in the second member of this inequality the quantity  $N$  studied above so that by inequality (7) we have

$$(10) \quad -\frac{\partial M}{\partial \rho} \leq \frac{\frac{1}{2} + k}{(\rho - k + \frac{1}{2})^2} + \frac{\frac{1}{2} - k}{(\rho - k - \frac{1}{2})^2} \leq \frac{1}{\rho(\rho - 1)}$$

for  $\rho \geq 3^{1/2}$ ,  $0 \leq k \leq \frac{1}{2}$ .

We have previously seen that the quantity

$$\rho - \frac{\frac{1}{2} + k}{(\rho - k + \frac{1}{2})^2} - \frac{\frac{1}{2} - k}{(\rho - k - \frac{1}{2})^2}$$

vanishes for  $\rho = \rho_0$  and is positive for all greater values of  $\rho$ . It follows that for all exterior motion we have

$$\rho > \frac{\frac{1}{2} + k}{(\rho - k + \frac{1}{2})^2} + \frac{\frac{1}{2} - k}{(\rho - k - \frac{1}{2})^2}$$

and this on combination with inequality (10) gives us

$$(11) \quad \rho + \frac{\partial M}{\partial \rho} > 0.$$

The other partial derivative of  $M$  with which we are chiefly concerned is

$$\frac{\partial M}{\partial \theta} = - \left(\frac{1}{4} - k^2\right)\rho \sin \theta \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right).$$

This evidently vanishes along the coördinate axes and is extremely small in absolute value for large values of  $\rho$ . To render this last statement more precise we observe from the figure that in quadrants I and IV we have

$$0 < \frac{1}{r_1^3} - \frac{1}{r_2^3} \leq \frac{1}{(\rho + k - \frac{1}{2})^3} - \frac{1}{(\rho + k + \frac{1}{2})^3}.$$

The derivative of the last member with respect to  $k$  being the negative quantity  $-3(\rho + k - \frac{1}{2})^{-4} + 3(\rho + k + \frac{1}{2})^{-4}$  that member must have its maximum value for  $k$  at the lower end of its range, i.e.,  $k=0$ . Hence we have

$$0 < \frac{1}{r_1^3} - \frac{1}{r_2^3} \leq \frac{1}{(\rho - \frac{1}{2})^3} - \frac{1}{(\rho + \frac{1}{2})^3} \text{ in I, IV.}$$

Similarly we find

$$0 < \frac{1}{r_2^3} - \frac{1}{r_1^3} \leq \frac{1}{(\rho - 1)^3} - \frac{1}{\rho^3} \text{ in II, III.}$$

Since the derivative of the function  $(\rho-1)^{-3} - \rho^{-3}$  ( $\rho > 1$ ) is the negative quantity  $-3(\rho-1)^{-4} + 3\rho^{-4}$  it follows that

$$\frac{1}{(\rho - \frac{1}{2})^3} - \frac{1}{(\rho + \frac{1}{2})^3} < \frac{1}{(\rho - 1)^3} - \frac{1}{\rho^3},$$

and we have in all four quadrants

$$\left| \frac{1}{r_1^3} - \frac{1}{r_2^3} \right| \leq \frac{1}{(\rho - 1)^3} - \frac{1}{\rho^3}.$$

Inspection of the inequality

$$-(\rho - 8)^3 - 15(\rho - 8)^2 - 53(\rho - 8) - 20 < 0$$

shows that it holds for  $\rho \geq 8$ . We may write this as

$$\frac{1}{(\rho - 1)^3} - \frac{1}{\rho^3} - \frac{4}{\rho^4} < 0 \quad (\rho \geq 8).$$

Hence

$$(12) \quad \left| \frac{\partial M}{\partial \theta} \right| = \left| \left(\frac{1}{4} - k^2\right)\rho \sin \theta \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) \right| < \frac{1}{\rho^3} \quad (\rho \geq 8).$$

We conclude this section by deriving two inequalities connecting the force function  $M$  and Jacobi's constant  $C$ . By the definition of  $\rho_1$  and  $C$  we have

$$\rho_1^2 + \frac{1+2k}{\rho_1 - k + \frac{1}{2}} + \frac{1-2k}{\rho_1 - k - \frac{1}{2}} = C,$$

or for  $\rho = \rho_1$

$$\frac{1+2k}{\rho - k + \frac{1}{2}} + \frac{1-2k}{\rho - k - \frac{1}{2}} = C - \rho_1^2.$$

Since  $\rho_1$  is the least value assumed by  $\rho$  for exterior motion and since any increase of  $\rho$  clearly causes a decrease in the first member of this equation it clearly follows that for all exterior motion we may write

$$\frac{1+2k}{\rho - k + \frac{1}{2}} + \frac{1-2k}{\rho - k - \frac{1}{2}} \leq C - \rho_1^2.$$

By inequality (6) this becomes

$$(13) \quad 2M \leq C - \rho_1^2.$$

Let us designate by  $A$  the value of the quantity  $\rho + \partial M / \partial \rho$  at the point where the closed outer oval crosses the negative  $X$  axis. By inequality (11)  $A$  is positive so that we have

$$\rho_1 - \frac{\frac{1}{2} + k}{(\rho_1 - k + \frac{1}{2})^2} - \frac{\frac{1}{2} - k}{(\rho_1 - k - \frac{1}{2})^2} = A > 0.$$

This definition is equivalent to saying that the equation

$$\frac{(\frac{1}{2} + k)\rho}{(\rho - k + \frac{1}{2})^2} + \frac{(\frac{1}{2} - k)\rho}{(\rho - k - \frac{1}{2})^2} = \rho_1^2 - A\rho_1$$

holds for  $\rho = \rho_1$ . Any increase of  $\rho$  causes a decrease in the left member of this equation and it follows that

$$\frac{(\frac{1}{2} + k)\rho}{(\rho - k + \frac{1}{2})^2} + \frac{(\frac{1}{2} - k)\rho}{(\rho - k - \frac{1}{2})^2} \leq \rho_1^2 - A\rho_1$$

for all exterior motion. By combining this with the first of inequalities (10) we obtain

$$- \rho \frac{\partial M}{\partial \rho} \leq \rho_1^2 - A\rho_1.$$

We add this member for member with inequality (13) obtaining

$$2M - C - \rho \frac{\partial M}{\partial \rho} \leq -A\rho_1,$$

and finally we recall that  $\rho_1$  is always greater than unity so that we may write

$$(14) \quad \frac{C - 2M}{\rho^2} + \frac{1}{\rho} \frac{\partial M}{\partial \rho} > \frac{A}{\rho^2} \quad \text{where } A > 0.$$

On the basis of the inequalities collected in this section we now proceed in the succeeding sections to study the exterior motion of the particle.

5. **Nature of the cusps.** In the rotating plane the function  $\Omega = \frac{1}{2}\rho^2 + M$  plays a rôle very similar to that of the force function  $M$  in fixed space. This may be brought out by putting the equations of motion in a suitable form and is evident also from the easily proved fact that if the particle be at rest in the moving plane on a curve  $2\Omega = C$  it will start its motion perpendicularly to this curve. It is, in fact, well known that the points of contact of an exterior orbit with the closed outer oval are at cusps of that orbit, the orbit being there orthogonal to the oval. With the aid of inequality (11) we may easily determine the species of these cusps in every case. Our equation (4) becomes at such a cusp

$$\rho\theta'' = -\left(\frac{1}{4} + k^2\right) \sin \theta \left( \frac{1}{r_1^3} - \frac{1}{r_2^3} \right).$$

For  $k \neq \frac{1}{2}$  this requires that  $\theta''$  be positive in the II and IV quadrants and negative in the I and III. So at such a cusp occurring in the II or IV quadrant,  $\theta'$  must change from negative to positive and the motion change from retrograde to direct relative to the moving plane. Similarly at a cusp occurring in the I or III quadrant the motion must change from direct to retrograde. For those cusps occurring on the coördinate axes and for the case  $k = \frac{1}{2}$ ,  $\theta''$  vanishes and we must examine  $\theta'''$ . Differentiating equation (4) with respect to the time gives us

$$\theta''' + 2\rho'\theta'' + 2\rho''(1 + \theta') = \left( \frac{1}{\rho} \frac{\partial M}{\partial \theta} \right)'$$

and since  $M$  is a point function at a cusp this becomes

$$\theta''' + 2\rho'' = 0.$$

But at a cusp equation (3) yields

$$\rho'' = \rho + \frac{\partial M}{\partial \rho}$$

and by inequality (11) the second member of this equation is always positive. Thus  $\rho'' > 0$  and therefore  $\theta''' < 0$  at such cusps and we have

$$\theta' = \theta'' = 0, \quad \theta''' < 0,$$

and these are the conditions that  $\theta'$  be at a maximum when it vanishes. Hence

$\theta' < 0$  both before and after such cusps. To summarize the situation in the moving plane in the neighborhood of the outer oval we may therefore state

**THEOREM III.** *The points of contact of exterior orbits with the closed outer oval are at cusps of these orbits, the orbit being there orthogonal to the oval. For  $k \neq \frac{1}{2}$  the motion is retrograde before and direct after such cusps as occur in the II or IV quadrants, and direct before and retrograde after such cusps as occur in the I and III quadrants. For  $k = \frac{1}{2}$  and for all cusps that occur on the coördinate axes the motion is retrograde both before and after the cusp.*

6. **Rejection to infinity.** In his paper (loc. cit. Theorem 3) Koopman devises a test giving sufficient conditions that an orbit recede to infinity. In the present section we shall present two other such tests applicable to exterior orbits. It will be observed that both these tests when satisfied yield orbits of the hyperbolic type, i.e. the velocity at infinity is positive.

**THEOREM IV.** *If at any moment in the motion of the infinitesimal body in an exterior orbit we have simultaneously*

$$(a) \quad \rho'^2 \geq 2M, \quad \rho' > 0,$$

*then from that moment on  $\rho$  increases continuously to infinity.*

The hypothesis  $\frac{1}{2}\rho'^2 \geq M$  may be visualized as a statement that the radial component of the specific kinetic energy of the particle exceeds or equals the potential due to the two finite masses.

Our hypothesis  $\rho'^2 \geq 2M$  and Jacobi's integral (5) lead at once to the inequality

$$(b) \quad \rho^2(1 - \theta'^2) \geq C$$

from which we may derive two conclusions. In the first place we have  $\rho \geq C^{1/2} \geq 3^{1/2}$  which enables us to employ inequality (9). Also we have from inequality (b)  $\theta'^2 \leq 1 - C/\rho^2$ , and consequently

$$\theta' \geq - \left( 1 - \frac{C}{\rho^2} \right)^{1/2} > -1 + \frac{C}{2\rho^2},$$

from which

$$(c) \quad \rho(1 + \theta')^2 > \frac{C^2}{4\rho^3} > \frac{2}{\rho^3}.$$

Equation (3) and inequality (10) and our inequality (c) now yield

$$\rho'' + \frac{\frac{1}{2} + k}{(\rho - k + \frac{1}{2})^2} + \frac{\frac{1}{2} - k}{(\rho - k - \frac{1}{2})^2} - \frac{2}{\rho^3} > 0.$$

Consequently the quantity

$$\frac{d}{dt} \left( \frac{\rho'^2}{2} - \frac{\frac{1}{2} + k}{\rho - k + \frac{1}{2}} - \frac{\frac{1}{2} - k}{\rho - k - \frac{1}{2}} + \frac{1}{\rho^2} \right)$$

has the sign of  $\rho'$  and it follows that in the function

$$P(t) \equiv \frac{\rho'^2}{2} - \frac{\frac{1}{2} + k}{\rho - k + \frac{1}{2}} - \frac{\frac{1}{2} - k}{\rho - k - \frac{1}{2}} + \frac{1}{\rho^2}$$

we have a quantity which increases and decreases with  $\rho$ . But it is apparent that if  $\rho' = 0$  then  $P(t) < 0$  since

$$(15) \quad \frac{\frac{1}{2} + k}{\rho - k + \frac{1}{2}} + \frac{\frac{1}{2} - k}{\rho - k - \frac{1}{2}} - \frac{1}{\rho} = \frac{\frac{1}{4} - k^2}{\rho(\rho - k + \frac{1}{2})(\rho - k - \frac{1}{2})} \geq 0$$

and consequently  $P'$  can vanish only when  $P$  is negative. Since we are dealing with an analytic function of  $t$  it follows at once that if at any instant  $t_0$  we have  $P \geq 0$  and increasing then it must thereafter remain positive and continually increasing. Now by hypothesis (a) and inequality (9) it is clear that the instant mentioned in the theorem is just such an instant, so that from the instant  $t_0$  on, both  $P$  and  $\rho$  continue to increase.

It remains to show that  $\rho$  becomes infinite. Let us define a function  $r(t)$  by the differential equation with boundary condition

$$(d) \quad r'(t) = \left( \frac{2}{3r} + 2P(t_0) \right)^{1/2}, \quad r(t_0) = \rho(t_0).$$

Then at the instant  $t_0$  we have

$$\frac{\rho'^2}{2} - \frac{\frac{1}{2} + k}{\rho - k + \frac{1}{2}} - \frac{\frac{1}{2} - k}{\rho - k - \frac{1}{2}} + \frac{1}{\rho^2} = \frac{r'^2}{2} - \frac{1}{3r},$$

and we know that the first member of this equation continues to increase after the instant  $t_0$  while by the definition of  $r(t)$  the second member remains constant. Hence we have

$$(e) \quad \frac{\rho'^2}{2} - \frac{\frac{1}{2} + k}{\rho - k + \frac{1}{2}} - \frac{\frac{1}{2} - k}{\rho - k - \frac{1}{2}} + \frac{1}{\rho^2} \geq \frac{r'^2}{2} - \frac{1}{3r} \quad (t \geq t_0).$$

But at no instant  $t > t_0$  can we have simultaneously  $\rho' < r'$ ,  $\rho < r$  because for  $\rho < r$  by inequality (15) and the fact that  $\rho \geq 3^{1/2}$  we would have

$$-\frac{\frac{1}{2} + k}{\rho - k + \frac{1}{2}} - \frac{\frac{1}{2} - k}{\rho - k - \frac{1}{2}} + \frac{1}{\rho^2} \leq -\frac{1}{\rho} + \frac{1}{\rho^2} < -\frac{1}{3\rho} < -\frac{1}{3r}$$

and inequality (e) would be violated. And now it also follows that at no

instant  $t_2 > t_0$  can we have  $\rho < r$ , for if  $\rho - r$  were negative at the instant  $t_2$  then there would exist some instant  $t_1$  ( $t_0 < t_1 < t_2$ ) at which both  $\rho - r$  and  $\rho' - r'$  would be negative; an impossible situation as just noted. We therefore have for  $t \geq t_0$ ,  $\rho \geq r$ . But the differential equation (d) is immediately solvable by elementary methods and shows that  $r$  becomes infinite with  $t$ . Hence  $\rho$  also becomes infinite and our theorem is proved.

Another set of conditions yielding orbits receding to infinity is given in the following

**THEOREM V.** *If at any moment in the motion of the infinitesimal body in an exterior orbit we have simultaneously*

$$\rho \geq C^{1/2}, \quad \theta' \geq 0, \quad \rho' \geq 0,$$

*then from that moment on  $\rho$  increases continuously to infinity.*

The argument of the proof is divided into two cases according to the value at the given moment of the quantity  $\rho'^2 - 2M$ .

**Case I.** If at the moment  $t_0$  when the conditions of our theorem are satisfied we have also  $\rho'^2 \geq 2M$ , then  $\rho' > 0$  and our theorem follows by Theorem IV.

**Case II.** If at the moment  $t_0$  we have also  $\rho'^2 < 2M$  then we shall show that there exists a moment  $t_1 > t_0$  at which  $\rho'^2 = 2M$  and such that  $\rho' > 0$  throughout the interval  $t_0 < t \leq t_1$ . Thus we shall see that  $\rho$  increases continuously throughout this interval and by application of Theorem IV to the instant  $t_1$  it will be evident that  $\rho$  increases continuously from the moment  $t_0$  and becomes infinite.

By equation (3) and inequality (11) we have  $\rho'' > 0$  whenever  $\theta' \geq 0$ . This condition is satisfied at the instant  $t_0$  and since  $\rho' \geq 0$  at that instant, it appears by the continuity of the quantities involved that there must exist some interval immediately subsequent to  $t_0$  during which  $\rho' > 0$  and  $\rho'^2 - 2M < 0$ . Thus either there exists an instant  $t_1$  such that

$$(a) \quad \begin{aligned} \rho'^2 - 2M < 0 \text{ for } t_0 \leq t < t_1 \text{ and } \rho'^2 - 2M = 0 \text{ for } t = t_1, \\ \text{or } \rho'^2 - 2M < 0 \text{ for } t \geq t_0. \end{aligned}$$

Also either there exists an instant  $t_2$  such that

$$(b) \quad \rho' > 0 \text{ for } t_0 < t < t_2 \text{ and } \rho' = 0 \text{ for } t = t_2, \text{ or } \rho' > 0 \text{ for } t \geq t_0.$$

We shall now establish that

(i) There exists no such instant  $t_2$  in the interval  $t_0 \leq t \leq t_1$ , nor in the interval  $t_0 \leq t$  in case  $t_1$  does not exist.

(ii) The instant  $t_1$  exists.



To establish the first statement we shall assume that such an instant  $t_2$  exists in the above interval  $t_0 \leq t \leq t_1$  or  $t_0 \leq t$  and show that this leads to a contradiction. We recall Jacobi's integral (5) and making use of the fact that  $\rho'^2 - 2M \leq 0$  we conclude

$$(c) \quad \theta'^2 \geq 1 - \frac{C}{\rho^2}.$$

At the instant  $t_0$  we have by hypothesis  $\rho^2 \geq C$  and since  $\rho' > 0$  for  $t_0 < t < t_2$  it follows that  $\rho^2 > C$  for  $t_0 < t \leq t_2$ , and therefore, by inequality (c),

$$(d) \quad \theta'^2 > 0 \text{ for } t_0 < t \leq t_2.$$

Now at the instant  $t_0$  we had by hypothesis  $\theta' \geq 0$ ,  $\rho \geq C^{1/2}$  and with the additional hypothesis  $\rho'^2 < 2M$  of Case II for that instant Jacobi's integral (5) shows us that the inequality  $\theta' > 0$  must hold. Thus by inequality (d) we have  $\theta' > 0$  for  $t_0 < t \leq t_2$  and since  $\rho'' > 0$  whenever  $\theta' > 0$  it follows that

$$(e) \quad \rho'' > 0 \text{ for } t_0 \leq t \leq t_2.$$

By the law of the mean this is a contradiction with the two values of  $\rho'$   $\rho'(t_0) \geq 0$ ,  $\rho'(t_2) = 0$ , and the first statement is proved.

To prove the existence of the instant  $t_1$  we shall assume that it does not exist and show that this leads to a contradiction. Under this assumption our first statement shows us that the instant  $t_2$  can not exist and the inequalities (a), (b), (e) must hold for  $t > t_0$ . By inequalities (b) and (e)  $\rho$  increases continuously to infinity from the instant  $t_0$ . Let  $t_3$  be some instant after  $t_0$  when  $\rho > 3^{1/2}$  and let  $h$  be the value of  $\rho'^2$  at that moment. Then by inequality (e) we have

$$(f) \quad \rho'^2 > h \text{ for } t > t_3,$$

while by inequality (8)

$$(g) \quad 2M < \frac{2}{\rho} + \frac{1}{2\rho^2} < \frac{7}{3\rho} \text{ for } t > t_3.$$

As soon after the instant  $t_3$  as  $\rho > 7/(3h)$  inequalities (f) and (g) contradict inequality (a) and our second statement is proved. The proof of the whole theorem is now also complete as outlined above.

We shall conclude this section by the statement of two lemmas. We do not prove them as theorems in this paper as here published as they are rather obvious extensions of facts already known. They constitute however necessary preliminaries to certain of the following theorems. The first lemma may be shown to be a consequence of our Theorem IV and Koopman's Theorem 5, loc. cit.

LEMMA I. *If at any moment in the motion of the infinitesimal body in an exterior orbit we have simultaneously  $\rho'^2 \geq 2M$ ,  $\rho' > 0$ , then as the infinitesimal body recedes to infinity its double areal velocity in fixed space  $\rho^2(1+\theta')$  approaches a limit  $G$ .*

Lemma II is an extension of Lemma I. It is designed to include orbits, if such there be, possessing infinitely many arcs extending outside of and returning within any circle  $\rho = R$  of arbitrarily great radius. It states in effect that  $\rho^2(1+\theta')$  approaches its limit uniformly for all the above arcs. More accurately put, this is

LEMMA II. *For any exterior orbit in which after a certain moment  $t_0$  the radial distance  $\rho$  does not remain finite there exists a constant  $G$  such that corresponding to any arbitrarily chosen positive number  $\epsilon$  we may find a positive constant  $R$  such that  $|\rho^2(1+\theta') - G| < \epsilon$  throughout every interval after  $t_0$  in which  $\rho$  exceeds  $R$ .*

7. **The angular velocity.** The present section is devoted to a study of the angular velocity of the infinitesimal body in exterior orbits relative to both the fixed and rotating planes. We shall say that the infinitesimal body is describing direct motion when it revolves about the center of mass of the two finite bodies in the same sense that they do. Motion in the opposite sense is called retrograde. Our first theorem requires little additional proof.

THEOREM VI. *In every exterior orbit the motion of the infinitesimal body about the center of mass is always direct with respect to fixed space and the angular velocity with respect to the rotating plane is always less than unity.*

Jacobi's integral (5) and our inequality (13) yield  $\rho'^2 + \rho^2\theta'^2 \leq \rho^2 - \rho_1^2$  and therefore we have  $\theta'^2 \leq 1 - \rho_1^2/\rho^2 - \rho'^2/\rho^2 < 1$ . If we write this as  $-1 < \theta' < +1$  we see that the angular velocity of the particle with respect to the rotating plane is always less than unity, thus proving the second part of the theorem. If we write our inequality as  $1 + \theta' > 0$  we see that the angular velocity  $1 + \theta'$  with respect to fixed space is always positive, thus proving the first part of the theorem.

The above theorem, being derived wholly from Jacobi's integral, bounds the angular velocity in the rotating plane as closely on the negative as on the positive side and might lead to the supposition that direct angular motion in the rotating plane were as general as retrograde. That this supposition is not correct is shown in the succeeding theorems of this section.

THEOREM VII. *Corresponding to any given exterior orbit and to any given instant  $t_0$  there exists a positive number  $R$  such that the motion of the infinitesimal*

body will be constantly retrograde in the rotating plane whenever after  $t_0$  the distance from the center of mass exceeds  $R$ .

We divide the proof into cases according to the values assumed by the quantity  $\rho'^2 - 2M$ . We first prove the theorem for Case I in which at some instant the conditions  $\rho'^2 \geq 2M$ ,  $\rho' > 0$  of Theorem IV are satisfied and in which therefore the particle recedes to infinity. Next in Case II we assume that the conditions of Case I never arise but that at some instant we have  $\rho'^2 \geq 2M$ ,  $\rho' < 0$ . We here show that these conditions always lead to a later instant at which the only remaining hypothesis  $\rho'^2 < 2M$  is satisfied and that the conditions of Case II can never recur. This leaves it only necessary to carry through the proof for the remaining Case III in which we do not at any instant have the conditions of Case I satisfied but do at some instant have  $\rho'^2 < 2M$ .

**Case I.** If at any instant  $t_1$  in the motion of the infinitesimal body in the given exterior orbit we have  $\rho'^2 \geq 2M$ ,  $\rho' > 0$ , then from that moment on  $\rho$  increases continuously to infinity by Theorem IV. According also to Lemma I we have

$$(a) \quad \lim_{\rho \rightarrow \infty} \rho^2(1 + \theta') = G.$$

If such an instant  $t_1$  occurs at or after the given instant  $t_0$ , then to establish our theorem we have merely to choose  $R$  greater than any value assumed by  $\rho$  in the closed interval  $t_0 \leq t \leq t_1$ , also greater than  $(G+h)^{1/2}$  and sufficiently great so that

$$(b) \quad |\rho^2(1 + \theta') - G| < h \text{ for } \rho > R,$$

the last being possible by equation (a) for  $h$  any positive constant. For with  $R$  so chosen whenever after  $t_0$  we have  $\rho > R$  we shall have by inequality (b)  $\rho^2(1 + \theta') < G + h$  and since  $\rho^2 > G + h$  it follows by subtraction that  $\rho^2\theta' < 0$ , proving our theorem for Case I.

**Case II.** If the conditions of Case I are not satisfied at any instant at or after  $t_0$  and if we have at some instant  $t_2 \geq t_0$  both  $\rho'^2 \geq 2M$ ,  $\rho' < 0$ , then there exists a moment  $t_3 > t_2$  such that

$$(c) \quad \rho'^2 < 2M \text{ for } t \geq t_3.$$

To prove this we observe as in the proof of Theorem IV that by inequality (9) the condition  $\rho'^2 \geq 2M$  requires that  $P(t) > 0$  for  $t = t_2$ . Also since  $\rho' < 0$  at that moment  $P(t)$  must be decreasing, and since  $P'(t)$  can only change sign for  $P(t) < 0$  it follows that either  $P(t)$  continues positive and decreasing or else becomes negative. But we may show that  $P(t)$  can not remain positive and

decreasing as this would yield two contradictory conclusions. In the first place, by the definition and properties of  $P(t)$  explained in the proof of Theorem IV and by inequalities (15) it is evident that as long as  $P(t)$  remains positive and decreasing we shall have

$$(d) \quad \rho' < - \left( \frac{1+2k}{\rho-k+\frac{1}{2}} + \frac{1-2k}{\rho-k-\frac{1}{2}} - \frac{2}{\rho^2} \right)^{1/2} < - \left( \frac{2}{\rho} - \frac{2}{\rho^2} \right)^{1/2},$$

and secondly from the fact that  $P(t) > 0$  together with Jacobi's integral (5) and inequality (6) we find  $\rho^2(1-\theta'^2) + 2/\rho^2 - C > 0$  and, a fortiori,  $\rho^2 + 2/\rho^2 > C \geq 3$ . Inspection of the equivalent inequality  $(\rho^2-1)(\rho^2-2) > 0$  shows that it is satisfied only for  $\rho^2 < 1$  or  $\rho^2 > 2$ . Hence if  $P(t)$  is to remain positive we must have  $\rho > 2^{1/2}$  and inequality (d) would now become

$$\rho' < - (2/\rho - 2/\rho^2)^{1/2} < - \left( \frac{2 - 2^{1/2}}{\rho(t_2)} \right)^{1/2}.$$

Thus we are led to the contradiction that  $\rho$  while remaining always greater than a constant continues to decrease at a rate greater than a positive constant. Thus  $P(t)$  becomes negative and consequently  $\rho'^2 - 2M$  must also become negative, since for  $\rho'^2 - 2M \geq 0$  we have  $P(t) > 0$ , as above noted. But  $\rho'^2 - 2M$  having thus become negative must remain so, for if at any subsequent instant  $t_4$  we had  $\rho'^2 - 2M = 0$  this would require that  $P(t)$  had previously become positive and at the instant  $t_5$  when  $P(t)$  thus changed from negative to positive we would have had  $P(t) = 0$ ,  $P'(t) \geq 0$ , and consequently  $P(t) = 0$ ,  $\rho' > 0$ . As shown in the proof of Theorem IV these last conditions enable us to write  $\rho' > 0$  for  $t \geq t_5$  and consequently at the instant  $t_4$  we would have simultaneously  $\rho'^2 = 2M$ ,  $\rho' > 0$ , contrary to our hypothesis that the conditions of Case I are not satisfied at or after  $t_0$ . This completes the proof that there exists an instant  $t_3$  such that condition (c) is satisfied. We shall show in the next paragraph that our theorem holds for the instant  $t_3$ . To insure that the theorem also holds for the instant  $t_0$  of the present paragraph we have merely to choose  $R$  as in the next paragraph for the instant  $t_3$  and then increase  $R$  if necessary so that it exceeds any value assumed by  $\rho$  in the closed interval  $t_0 \leq t \leq t_3$ .

**Case III.** It remains only to consider the case  $\rho'^2 < 2M$  for  $t \geq t_0$ . We first observe that under this hypothesis Jacobi's integral (5) yields the inequality  $\theta'^2 > 1 - C/\rho^2$  so that we have  $\theta' \neq 0$  for  $\rho \geq C^{1/2}$  and  $t \geq t_0$ . We can now prove without difficulty that for the number  $R$  we have merely to choose a number larger than  $C^{1/2}$  and larger than the value of  $\rho$  at the instant  $t_0$ . Since  $\rho < R$  for  $t = t_0$  we must have  $\rho' \geq 0$  at the beginning of each interval in which  $\rho > R$ . Also if  $\theta' > 0$  at any moment during any such interval we would have  $\theta' > 0$

throughout the interval. But this is impossible, for if so at the beginning of the interval we would have simultaneously  $\rho'^2 < 2M$ ,  $\rho > C^{1/2}$ ,  $\rho' \geq 0$ ,  $\theta' > 0$ , and under these conditions we demonstrated in Case II of the proof of Theorem V that there exists a later moment at which  $\rho'^2 = 2M$ , contrary to our present hypothesis. Thus  $\theta' < 0$  throughout any interval in which  $\rho$  exceeds  $R$  after the instant  $t_0$ , and the theorem is proved for the third and last case.

Theorem VII shows us that after the instant  $t_0$  the arcs of the orbit extending outside of the circle  $\rho = R$  are everywhere retrograde in the rotating plane. In certain orbits, however, this theorem might be without application since  $\rho$  might never exceed  $R$ . Nevertheless we may very easily show that in such a case also the outer arcs of the orbit are retrograde in the rotating plane.

**THEOREM VIII.** *The motion of the infinitesimal body in an exterior orbit is retrograde in the rotating plane in the neighborhood of every point where  $\rho$  passes through a maximum value.*

For such a maximum value of  $\rho$  we have  $\rho'' \leq 0$ . But equation (4) and inequality (11) yield the inequality  $\rho'' > \rho(2\theta' + \theta'^2)$  showing that at any maximum of  $\rho$  we must have  $\theta' < 0$ . Also due to its continuity  $\theta'$  must remain negative throughout some neighborhood of such points, as stated in the theorem.

Theorems VII and VIII place considerable restriction on the direct motion of the infinitesimal body relative to the rotating plane. But if we confine our attention to those orbits, such as the periodic ones, in which the radial distance  $\rho$  remains finite, we may very materially strengthen this restriction. In fact after a certain moment the direct motion may take place only within a certain ring about the closed outer oval and the ring is very narrow for large values of Jacobi's constant  $C$ .

**THEOREM IX.** *For each exterior orbit for which the radial distance  $\rho$  remains finite after a certain moment there exists an instant  $t_0$  such that after that instant the motion can be direct in the rotating plane only within the ring  $\rho_1 < \rho < C^{1/2}$  ( $C$  and  $\rho_1$  being defined as in §3). If we take a sequence of orbits having values of  $C$ :  $C_1, C_2, C_3, \dots$  such that  $C$  becomes infinite on the sequence, the width of the above ring approaches zero on the sequence, its principal part being  $1/C$ .*

The proof for the first part of the theorem will be presented in the following steps:

- I. For every exterior orbit in which  $\rho$  remains finite after a certain moment we shall show that there exists an instant  $t_1$  such that  $P(t) < 0$  for  $t > t_1$ .
- II. If there is no instant after  $t_1$  at which both  $\rho \geq C^{1/2}$ ,  $\theta' \geq 0$ , then our theorem is granted for this orbit,  $t_1$  becoming the instant  $t_0$  of the theorem.

III. If there is an instant  $t_2 > t_1$  at which both  $\rho \geq C^{1/2}$ ,  $\theta' \geq 0$ , we shall show that there is an instant  $t_3 > t_2$  at which  $\rho < C^{1/2}$ .

IV. We shall show that for  $t > t_3$  we can never have both  $\rho \geq C^{1/2}$ ,  $\theta' \geq 0$ ,  $t_3$  becoming the instant  $t_0$  of the theorem.

I. If  $P(t)$  does not become positive, step I is granted at once. But if  $P(t) \geq 0$  at any instant then  $\rho' < 0$  at that instant, for as seen by inequality (15)  $\rho'$  can not then be zero, and  $\rho'$  can not be positive for we saw in the proof of Theorem IV that if there exists any moment at which both  $P(t) \geq 0$ ,  $\rho' > 0$ , then  $\rho$  becomes infinite, contrary to our present hypothesis. But we showed in Case II of the proof of Theorem VII that the condition  $P(t) \geq 0$ ,  $\rho' < 0$ , can not persist and there must be a later moment  $t_1$  at which  $P(t) < 0$ . But  $P(t)$  having once become negative can not again become positive or zero. For if  $P(t)$  were thus to increase to or through the value zero we would at that moment have both  $P(t) = 0$ ,  $P'(t) \geq 0$ , and consequently also  $\rho' > 0$ , which combination as previously noted is impossible. This establishes step I, and step II needs no further explanation. In steps III and IV we shall be assuming  $P(t) < 0$ .

III. If  $\rho$  remains finite and there exists an instant  $t_2$  such that  $\rho \geq C^{1/2}$ ,  $\theta' \geq 0$  for  $t = t_2$  and  $P(t) < 0$  for  $t \geq t_2$ , then there exists a moment  $t_3 > t_2$  at which  $\rho < C^{1/2}$ . We shall assume that the instant  $t_3$  does not exist and show that this leads to a contradiction. Since  $P(t) < 0$  we have also  $\rho'^2 < 2M$ , as previously remarked, and hence by Jacobi's integral (5)  $\theta'^2 > 1 - C/\rho^2$  for  $t \geq t_2$ . But we are now assuming that  $\rho \geq C^{1/2}$  and so by the above inequality  $\theta' \neq 0$  and since we had by hypothesis  $\theta' \geq 0$  for  $t = t_2$  we must have  $\theta' > 0$  for  $t \geq t_2$ . This fact together with equation (3) and inequality (10) yields at once the conclusion  $\rho'' \geq \rho - 1/[\rho(\rho - 1)]$  for  $t \geq t_2$  and since we have  $\rho \geq C^{1/2} \geq 3^{1/2}$  this becomes  $\rho'' > 9/10$ . On the other hand we must have  $\rho' < 0$  for  $t \geq t_2$  since for  $\rho' \geq 0$  in this interval we would have simultaneously  $\rho \geq C^{1/2}$ ,  $\rho' \geq 0$ ,  $\theta' > 0$ , and this by Theorem V would require that  $\rho$  become infinite, contrary to hypothesis. Thus the supposition that there exists no moment  $t_3 > t_2$  at which  $\rho < C^{1/2}$  leads to the contradictory conclusions  $\rho'' > 9/10$ ,  $\rho' < 0$  for every  $t \geq t_2$ , and the instant  $t_3$  must exist.

IV. We may now easily complete the proof of the first part of the theorem by showing that for any exterior orbit in which  $\rho$  remains finite and for which there exists an instant  $t_3$  such that  $\rho < C^{1/2}$  for  $t = t_3$  and  $P(t) < 0$  for  $t \geq t_3$ , then there can exist no later instant at which both  $\rho \geq C^{1/2}$ ,  $\theta' \geq 0$ . We first recall that under our hypotheses  $\theta' \neq 0$  for  $\rho \geq C^{1/2}$  so that if  $\theta' > 0$  at any moment in an interval throughout which  $\rho \geq C^{1/2}$ ,  $\theta' > 0$  throughout that interval. Since  $\rho < C^{1/2}$  at the instant  $t_3$ , if we are later to have  $\rho \geq C^{1/2}$  there must be some instant at which  $\rho = C^{1/2}$ ,  $\rho' \geq 0$  constituting the beginning of the interval

for which  $\rho \geq C^{1/2}$ . But now if we had  $\theta' > 0$  at any moment in such an interval we would have  $\theta' > 0$  throughout the interval. Hence we would have simultaneously at the beginning of the interval  $\rho = C^{1/2}$ ,  $\rho' \geq 0$ ,  $\theta' > 0$ , and this is impossible since by Theorem V this would require that  $\rho$  become infinite, contrary to hypothesis. The first part of the theorem is thus proved.

It remains to consider the width of this ring  $\rho_1 < \rho < C^{1/2}$  in which all direct motion must take place. We first recall from §3 the equation

$$\rho_1^2 + \frac{1 + 2k}{\rho_1 - k + \frac{1}{2}} + \frac{1 - 2k}{\rho_1 - k - \frac{1}{2}} = C.$$

Since  $C$  is here a continuous function of  $\rho_1$  with non-vanishing derivative in the interval  $\rho_1 > \rho_0$  it follows that  $\rho_1$  is likewise a continuous function of  $C$  in the corresponding interval  $C > C_0$  and  $\rho_1$  becomes infinite with  $C$ . It is evident that

$$\lim_{\rho_1 \rightarrow \infty} \frac{C}{\rho_1^2} = 1,$$

and from these last two equations it follows that

$$\lim_{C \rightarrow \infty} \frac{C^{1/2} - \rho_1}{1/C} = 1.$$

Thus we see that the width  $C^{1/2} - \rho_1$  of the ring of the theorem approaches zero as  $C$  becomes infinite, its principal part being  $1/C$ .

**8. The angular displacement.** We have seen in the four theorems of §7 certain conclusions that may be reached concerning the angular velocity of the infinitesimal body in an exterior orbit. The present group of three theorems draws somewhat analogous conclusions concerning the total angular displacement without concern as to the rate at which it is performed.

**THEOREM X.** *In every exterior orbit after any given instant  $t_0$  the infinitesimal body performs infinitely many retrograde circuits in the rotating plane about the two finite masses.*

We shall divide the proof of our theorem into Case I in which  $\rho$  remains finite and Case II in which  $\rho$  does not remain finite. For Case I our equations (3) and (5) give us

$$\frac{\rho''}{\rho} - \frac{\rho'^2}{\rho^2} = 2(\theta' + \theta'^2) + \frac{C - 2M}{\rho^2} + \frac{1}{\rho} \frac{\partial M}{\partial \rho},$$

and if we introduce a new variable  $\xi = \rho'/\rho$  into equation (5) and this last equation they take the forms

$$\xi^2 = 1 - \theta'^2 - (C - 2M)/\rho^2,$$

$$\xi' = 2(\theta' + \theta'^2) + \frac{C - 2M}{\rho^2} + \frac{1}{\rho} \frac{\partial M}{\partial \rho}.$$

By inequalities (13) and (14) these become

$$(16) \quad \xi^2 \leq 1 - \theta'^2 - \rho_1^2/\rho^2,$$

$$(17) \quad \xi' > 2(\theta' + \theta'^2) + A/\rho^2 \quad (A > 0).$$

An immediate consequence of inequality (16) is that

$$(18) \quad -1 < \xi < +1.$$

Under the hypothesis of Case I there exists a positive number  $R$  such that  $\rho < R$  for  $t \geq t_0$  so that inequality (17) gives us  $\xi' > 2(\theta' + \theta'^2) + A/R^2$  and a fortiori  $-\theta' < -\frac{1}{2}\xi' + A/(2R^2)$  for  $t \geq t_0$ . Now if  $t_1$  be any instant after the given instant  $t_0$  and if  $\theta_0, \xi_0, \theta_1, \xi_1$  be the values of  $\theta$  and  $\xi$  at these instants, we shall have

$$-\int_{t_0}^{t_1} \theta' dt > -\frac{1}{2} \int_{t_0}^{t_1} \xi' dt + \int_{t_0}^{t_1} \frac{A}{2R^2} dt$$

or

$$-(\theta_1 - \theta_0) > \frac{1}{2}(\xi_0 - \xi_1) + \frac{A}{2R^2}(t_1 - t_0).$$

By inequalities (18) it is evident that  $1 > \frac{1}{2}(\xi_1 - \xi_0)$  and hence

$$-(\theta_1 - \theta_0) > \frac{A}{2R^2}(t_1 - t_0) - 1.$$

Now being given an arbitrary positive number  $n$  we may choose  $t_1 = t_0 + 2R^2 \cdot (2\pi n + 1)/A$ , and our last inequality will yield

$$-(\theta_1 - \theta_0) > 2\pi n,$$

thus showing that in the time interval  $t_1$  to  $t_0$  the infinitesimal body performed at least  $n$  retrograde circuits about the center of mass. Since  $n$  is arbitrary the theorem follows for Case I.

We consider now Case II in which  $\rho$  does not remain finite. By Theorem VII we may find a positive number  $R_1$  such that  $\theta' < 0$  whenever  $\rho > R_1$ . Now if  $h$  and  $k$  be any two positive numbers ( $k < 1$ ) we may by Lemma I find a positive number  $R_2$  such that

$$(a) \quad \rho^2(1 + \theta') < G + h \text{ for } \rho > R_2,$$

and we may set



$$(b) \quad \left( \frac{G+h}{1-k} \right)^{1/2} = R_3.$$

Now if  $R$  is any number larger than  $R_1, R_2, R_3$ , then, after the given instant  $t_0$ ,  $\rho$  must satisfy for an infinite duration of time one or the other or both of the two inequalities

$$(c) \quad \rho \leq R, \quad (d) \quad \rho \geq R.$$

It is of course not implied that this duration of time is consecutive. If the inequality (c) be satisfied for an infinite duration of time we may apply the proof of Case I of our theorem, it being merely necessary to replace the true time  $t$  by a fictitious time  $\tau$  consisting of those intervals of the true time after  $t_0$  during which inequality (c) is satisfied. We thus find that the infinitesimal body completes an infinite number of retrograde circuits during the time that  $\rho \leq R$ . Since  $\theta' < 0$  for  $\rho > R$  the presence of such intervals can not invalidate the conclusion. But on the other hand if inequality (c) is satisfied for only a finite duration of time after  $t_0$  then since  $|\theta'| < 1$  the total angular displacement during this time must be finite and we may indicate it by  $\psi$ . But the duration of time after  $t_0$  during which inequality (d) is satisfied must now be infinite and we may introduce a new fictitious time  $\tau$  consisting of these intervals of the true time after  $t_0$  for which we have (d). Since inequality (a) holds throughout  $\tau$  and since we have by equation (b)  $\rho^2 > (G+h)/(1-k)$  it follows by division of these two inequalities that  $\theta' < -k$ . Now let  $\tau_1$  and  $\tau_2$  ( $\tau_2 > \tau_1$ ) be any two values of  $\tau$  and let  $\Delta\theta$  be the increment of  $\theta$  during the  $\tau$  interval  $\tau_1 \leq \tau \leq \tau_2$ . Then we have

$$- \int_{\tau_1}^{\tau_2} \theta' d\tau > \int_{\tau_1}^{\tau_2} k d\tau$$

or

$$- \Delta\theta > k(\tau_2 - \tau_1).$$

Now being given an arbitrary positive number  $n$  let us fix  $\tau_1$  and then choose  $\tau_2 = \tau_1 + (2\pi n + \psi)/k$ . Our last inequality then becomes  $-\Delta\theta > 2\pi n + \psi$  showing that in the true time interval from  $t_0$  to  $\tau_2$  the body performs at least  $n$  retrograde circuits, it being observed that all motion in the  $\tau$  interval previous to  $\tau_1$  is also retrograde. The theorem is thus established.

The next theorem has to do with motion of the infinitesimal body in fixed space. Let us set up in the plane of the motion a polar coördinate system fixed in space having its initial line and pole coinciding with the position occupied by those for the rotating plane at some instant  $t=0$ . Then the new vectorial angle  $\phi = \theta + t$  and  $\phi' = \theta' + 1$ . We may now prove

**THEOREM XI.** *In every exterior orbit in which the infinitesimal body remains after a certain instant at a finite distance from the center of mass of the system it will thereafter perform infinitely many direct circuits in fixed space about the two finite masses.*

By Jacobi's integral (5) and inequality (13) we have  $\rho^2(1-\theta'^2) \geq \rho_1^2 + \rho'^2 \geq \rho_1^2$ . Also by Theorem VI we saw that  $-1 < \theta' < +1$  or  $0 < 1-\theta' < +2$  and these give  $\rho^2(1+\theta') > \rho_1^2/2$ . Now by hypothesis there exists a positive constant  $R$  such that  $\rho < R$  for  $t > t_0$  and we therefore strengthen our last inequality when we write  $\phi' = 1+\theta' > \rho_1^2/(2R^2)$ . If  $t_1$  be any instant after  $t_0$  and if  $\phi_1$  and  $\phi_0$  be the corresponding values of  $\phi$ , then by this inequality we have

$$\int_{t_0}^{t_1} \phi' dt > \int_{t_0}^{t_1} \frac{\rho_1^2}{2R^2} dt \quad \text{or} \quad \phi_1 - \phi_0 > \frac{\rho_1^2}{2R^2}(t_1 - t_0).$$

Now being given an arbitrary positive number  $n$ , we may choose  $t_1 = t_0 + 4\pi n R^2/\rho_1^2$  and our last inequality becomes  $\phi_1 - \phi_0 > 2\pi n$ , thus showing that in the interval from  $t_0$  to  $t_1$  the infinitesimal body performed at least  $n$  direct circuits in fixed space. Since  $n$  is arbitrary the theorem follows.

We have seen that the retrograde motion of the infinitesimal body in the rotating plane tends to exceed the direct by unlimitedly large amounts in sufficiently long intervals of time. We further show in our next theorem that the direct motion can not much exceed the retrograde in *any* interval of time.

**THEOREM XII.** *In no portion of any exterior orbit can the direct angular motion in the rotating plane exceed the retrograde by as much as one radian.*

Thus if the infinitesimal body enters a sector  $FGH$  of angle one radian and vertex at the center of mass  $G$  across its initial side  $FG$ , then it must next leave that sector back again across the same side  $FG$ . We strengthen inequality (17) by writing  $\theta' < \frac{1}{2}\xi'$ . Now let  $t_1$  and  $t_2$  ( $t_2 > t_1$ ) be any two instants and let  $\theta_1, \xi_1, \theta_2, \xi_2$  be the corresponding values of  $\theta$  and  $\xi$ . Then by our last inequality and by (18) we have

$$\theta_2 - \theta_1 = \int_{t_1}^{t_2} \theta' dt < \frac{1}{2} \int_{t_1}^{t_2} \xi' dt = \frac{1}{2}(\xi_2 - \xi_1) < 1,$$

as we wished to prove. The implication in the theorem that the infinitesimal body must leave the sector  $FGH$  is of course justified by Theorem X.

Under more restricted hypotheses the conclusion of the above theorem may be considerably strengthened as shown in the following corollaries.

**COROLLARY I.** *Under the hypotheses of Theorem IV or Theorem V and after the given instant the direct angular motion of the infinitesimal body in the rotating*

plane may in no portion of the orbit exceed the retrograde by as much as  $\frac{1}{2}$  a radian.

As proved in Theorem IV and Theorem V the quantity  $\rho'$  is positive from the given moment on. Hence our inequality (18) may be strengthened to read  $0 < \xi < +1$  and consequently  $\frac{1}{2} > \frac{1}{2}(\xi_2 - \xi_1)$ . Since we still have  $\theta_2 - \theta_1 < \frac{1}{2}(\xi_2 - \xi_1)$  we conclude at once that  $\theta_2 - \theta_1 < \frac{1}{2}$  as we wished to prove.

**COROLLARY II.** *Under the hypotheses of Theorem IX and on the sequence of orbits there defined, the angle  $FGH$  of the sector of Theorem XII may be so chosen as to approach zero with a principal part  $2^{1/2}C^{-3/4}$ .*

In the proof of Theorem IX we saw that after a certain moment we have  $P(t) < 0$ . This together with the fact that  $\rho \geq \rho_1$  enables us to write

$$\xi^2 < \frac{1}{\rho^2} \left( \frac{1 + 2k}{\rho - k + \frac{1}{2}} + \frac{1 - 2k}{\rho - k - \frac{1}{2}} \right) \leq K^2$$

where

$$K = \frac{1}{\rho_1} \left( \frac{1 + 2k}{\rho_1 - k + \frac{1}{2}} + \frac{1 - 2k}{\rho_1 - k - \frac{1}{2}} \right)^{1/2}.$$

Now following the method and notation of the proof of Theorem XII we see that here  $\frac{1}{2}(\xi_2 - \xi_1) < K$  and since  $\theta_2 - \theta_1 < \frac{1}{2}(\xi_2 - \xi_1)$  we have at once  $\theta_2 - \theta_1 < K$ . Thus the angle of the sector of Theorem XII need never exceed  $K$  after a certain instant,  $K$  being a constant for the orbit. If we take the sequence of orbits of Theorem IX having  $C$  and  $\rho_1$  becoming infinite on the sequence, then the following equations hold:

$$\lim_{\rho_1 \rightarrow \infty} \frac{C}{\rho_1^2} = 1, \quad \lim_{\rho_1 \rightarrow \infty} K^2 \rho_1^3 = 2,$$

so that we have at once

$$\lim_{C \rightarrow \infty} \frac{K}{2^{1/2}C^{-3/4}} = 1.$$

Since the angle of the sector  $FGH$  may be chosen equal to  $K$  the corollary is now established.

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